Math 103: Final Exam

Spring 2006

- You have two hours and thirty minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

Problem 1. Let A, B, C be nonempty sets, $f : A \to B$ and $g : B \to C$ be functions, and for any function h denote by "Dom(h)" the domain of h. Use the definition of domain and inverse image to prove that $\text{Dom}(g \circ f) = f^{-1}(\text{Dom}(g))$, i.e., domain of $g \circ f$ is the inverse image of the domain of g under f. (15 points)

Problem 2. Prove that the set $A := (0, 1) - \mathbb{Q}$ of irrational numbers belonging to the interval $(0, 1)$ is uncountable. (10 points)

Problem 3. We have proven in class that every nonnegative integer n admits a binary expansion, i.e., $\forall n \in \mathbb{N}$, $\forall i \in \mathbb{N}$, $\exists\, a_i \in \{0, 1\}$, such that $n = \sum_{i=0}^{\infty} a_i 2^i$, where clearly only finitely many of the coefficients a_i can be 1. Prove using induction that the binary expansion of n is unique, i.e., show that " $\forall i \in \mathbb{N}$, $\exists\, a_i, b_i \in \{0, 1\}$, such that $\sum_{i=0}^{\infty} a_i 2^i = \sum_{i=0}^{\infty} b_i 2^i$ " implies " $\forall i \in \mathbb{N}$, $a_i = b_i$. " (15 points)

Problem 4. Prove that the set $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is countable. (15 points)
Hint: Construct a one-to-one function $f : \mathcal{F} \to \mathbb{N}$ with domain $\mathcal{F}$ by mapping finite subsets A of $\mathbb{N}$ onto the binary expansion of natural numbers n.

Problem 5. Let S be a set and 2^S be its power set. Let C denote the set of cardinal numbers of the elements of 2^S and define the relation $\leq$ on C according to:

$$\alpha \leq \beta \;\text{ if }\; (\exists A \in \alpha, \exists B \in \beta, \exists \text{ a one-to-one function } f : A \to B \text{ with domain A}).$$

5.a) Prove that $\alpha \leq \beta$ implies

$$\forall S \in \alpha, \forall T \in \beta, \exists \text{ a one-to-one function } g : S \to T \text{ with domain S.} \qquad (10 \text{ points})$$

5.b) Prove that $\leq$ is a partial ordering relation on C. (10 points)

Problem 6. Let $\mathcal{U} := \{S \mid S \text{ is a set.}\}$. Prove that $\mathcal{U}$ is not a set. (15 points)

Problem 7. Let A be the subset of $\mathbb{R}$ defined by

$$A := \left\{ x \in \mathbb{R} \mid \forall n \in \mathbb{Z}^+, 0 \leq x < \frac{1}{n} \right\}.$$

7.a) Prove that A has a least upper bound b in $\mathbb{R}$. (3 points)

7.b) Prove that $b = 0$. (6 points)
Hint: You may use the statement that $\forall a \in \mathbb{R}^+$, $\exists n \in \mathbb{Z}^+$, $\frac{1}{n} < a$.

7.c) Prove that $A = \{0\}$. (1 point)