Phys 312: Quiz # 2

Fall 2019

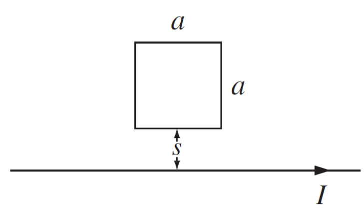
- <u>Write your name</u> and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have <u>One hour</u>.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

* *

1 (12 points) A square loop of wire (of side length a) lies on a table, a distance s from a very long straight wire, which carries a constant current I, as shown below. The loop is pulled away from the loop with a constant speed v. Find the generated emf and the direction of the induced current in the loop.



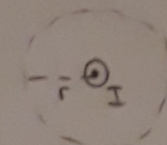
2 (18 points) Consider an electromagnetic field $(\mathbf{E}, \mathbf{B})$ in a homogeneous, isotropic, linear dielectric medium with permittivity ε and permeability μ. Suppose that this medium does not include any free charges or currents, and that there are a real parameter ω and vector-valued functions of space $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ such that $\mathbf{E}(\mathbf{r}, t) = \cos(\omega t)\boldsymbol{\mathcal{E}}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r}, t) = \sin(\omega t)\boldsymbol{\mathcal{B}}(\mathbf{r})$. Let $\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{k})$ be the Fourier transform of $\boldsymbol{\mathcal{E}}(\mathbf{r})$, i.e., $\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{k}) := \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\mathcal{E}}(\mathbf{r}) d^3\mathbf{r}$. Use Maxwell's equations to show that $\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{k}) = \mathbf{0}$ for $|\mathbf{k}| \neq \sqrt{\mu\varepsilon}\,\omega$.

Note: You may use the following identity: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

QUIZ 2 - SOLUTIONS

1) Ampere law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

Choose the following Amperian loop



By symmetry, we expect $\vec{B}$ in $\hat{\varphi}$ direction in polar coords.
Then the Ampere law becomes $B\, 2\pi r = \mu_0 I$

$$\Longrightarrow \quad \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

Magnetic flux over a surface S is given by

$$\phi_B := \int_S \vec{B}\cdot d\vec{a}$$

Choose S as the region enclosed by the square loop which is moving with a speed v. Then

$$\phi_B = \int_S \frac{\mu_0 I}{2\pi s}\, da \qquad \left(\text{because } \vec{B} \parallel d\vec{a}\right)$$

where s is the time dependent distance of the lower edge to the straight wire.

Note that $da = a\, ds$, so we have

$$\phi_B = \frac{\mu_0 I a}{2\pi}\int_s^{s+a}\frac{ds}{s} = \frac{\mu_0 I a}{2\pi}\left[\log(s+a) - \log s\right]$$

Emf is given by $\mathcal{E} = -\frac{d\phi_B}{dt}$ Hence

$$\mathcal{E} = -\frac{d}{dt}\left\{\frac{\mu_0 I a}{2\pi}\left[\log(s+a) - \log s\right]\right\}$$

$$= -\frac{\mu_0 I a}{2\pi}\left(\frac{1}{s+a}\frac{ds}{dt} - \frac{1}{s}\frac{ds}{dt}\right)$$

$$= -\frac{\mu_0 I a v}{2\pi}\left(\frac{1}{s+a} - \frac{1}{s}\right) = \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

To cancel the effect of the motion (Lenz law) the current should flow counterclockwise.

2) $\vec{E}(\vec{r},t) = \cos(\omega t)\,\vec{\mathcal{E}}(\vec{r})$

$\vec{B}(\vec{r},t) = \sin(\omega t)\,\vec{\mathcal{B}}(\vec{r})$

$$\tilde{\vec{\mathcal{E}}}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\vec{k}\cdot\vec{r}}\,\vec{\mathcal{E}}(\vec{r})\,d^3\vec{r}$$

Show $\tilde{\vec{\mathcal{E}}}(\vec{k}) = 0$ for $|\vec{k}| \neq \sqrt{\mu\varepsilon}\,\omega$

We have $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{B} = \mu\varepsilon\frac{\partial \vec{E}}{\partial t}$

$$\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = \frac{\partial}{\partial t}\left(\mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right) \Rightarrow \vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \vec{\nabla}\times(-\vec{\nabla}\times\vec{E}) = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow -\underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{=0 \text{ by Gauss law (no free charge)}} + \nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \cos(\omega t)\,\nabla^2\vec{\mathcal{E}} = -\mu\varepsilon\omega^2\cos(\omega t)\,\vec{\mathcal{E}}$$

$$\Rightarrow \nabla^2\vec{\mathcal{E}} = -\mu\varepsilon\omega^2\vec{\mathcal{E}}$$

The general solution to this eq$^{\underline{n}}$ is given by

$$\vec{\mathcal{E}}(\vec{r}) = A e^{i\vec{k}'\cdot\vec{r}} + B e^{-i\vec{k}'\cdot\vec{r}}$$

where $|\vec{k}'|^2 = \mu\varepsilon\omega^2$, A, B are constants

Now, insert this into the Fourier transform

$$\tilde{\mathcal{E}}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\vec{k}\cdot\vec{r}} \left(A e^{i\vec{k}'\cdot\vec{r}} + B e^{-i\vec{k}'\cdot\vec{r}} \right) d^3\vec{r}$$

$$= \frac{1}{(2\pi)^{3/2}} \left[A \underbrace{\int_{\mathbb{R}^3} e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} d^3\vec{r}}_{\delta^{(3)}(\vec{k}-\vec{k}')} + B \underbrace{\int_{\mathbb{R}^3} e^{-i(\vec{k}+\vec{k}')\cdot\vec{r}} d^3 r}_{\delta^{(3)}(\vec{k}+\vec{k}')} \right]$$

Hence $\tilde{\mathcal{E}}(\vec{k})$ is non-zero only when $\vec{k} = \pm \vec{k}'$ and $|\vec{k}'|^2 = \mu \mathcal{E} \omega^2$. Hence $\tilde{\mathcal{E}}(\vec{k})$ is non-zero for $|\vec{k}| \neq \sqrt{\mu\mathcal{E}}\, \omega$.