Phys 312: Quiz # 1

Fall 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	Solutions

- You have One hour.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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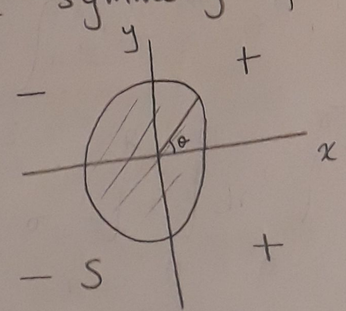
1 (10 points) Let (r, ϑ, z) be the cylindrical coordinates in plane, R and σ_0 be positive real parameters, and D be the disc given by $r \leq R$ and $z = 0$. Find the electric dipole moment of D if its charge density has the form $\sigma = \sigma_0 \cos\vartheta$.

Dipole moment of a surface charge density σ over a surface S

$$\vec{p} = \int_S \vec{r}\,'\sigma(\vec{r}\,')\,da'$$

We need to work with the polar coordinates.

By the symmetry of the charge density $\sigma_0 \cos\theta$, we have.



So we expect to have a polarization in the direction $\hat{x}$

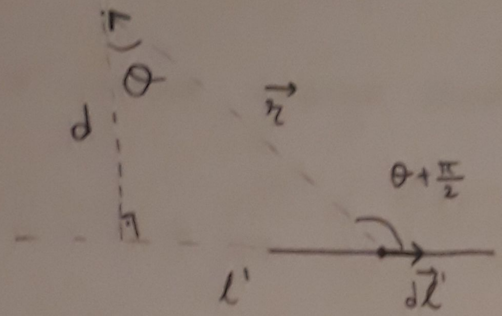
Hence we focus only on x-components

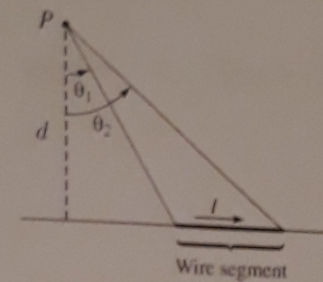
$$p = \int_S r'\cos\theta'\,\sigma_0\cos\theta'\,r'\,dr'\,d\theta'$$

because the Jacobian is r'

$$= \sigma_0 \int_{r'=0}^{R} (r')^2\,dr' \int_{\theta'=0}^{2\pi} (\cos\theta')^2\,d\theta' = \frac{\sigma_0 R^3 \pi}{3}$$

2 (10 points) Find the magnetic field **B** due to a segment of a long straight wire carrying a steady current I at a point P that is a distance d away from the wire. Express **B** in terms of I, d, and the angles θ_1 and θ_2 that are shown in the following figure.





$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\vec{\eta}}}{\eta^2}$$ (Biot-Savart law)

For our problem,

$$l' = d\tan\theta \Rightarrow dl' = \frac{d}{\cos^2\theta}\,d\theta$$

$$\cos\theta = \frac{d}{\eta} \Rightarrow \frac{1}{\eta^2} = \frac{\cos^2\theta}{d^2}$$

$$|d\vec{l}' \times \hat{\vec{\eta}}| = dl'\sin\left(\theta + \frac{\pi}{2}\right) = \frac{d}{\cos^2\theta}\cos\theta\,d\theta = \frac{d}{\cos\theta}\,d\theta$$

The direction is $\odot$ by right-hand rule.

$$|\vec{B}(\vec{r})| = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{d}{\cos\theta}\,\frac{\cos^2\theta}{d^2}\,d\theta = \frac{\mu_0 I}{4\pi d}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta$$

$$= \frac{\mu_0 I}{4\pi d}\,\sin\theta\,\Big|_{\theta=\theta_1}^{\theta_2} = \frac{\mu_0 I}{4\pi d}(\sin\theta_2 - \sin\theta_1)$$

The direction is $\odot$

3 (10 points) Let **B** be the magnetic field due to a continuous distribution of currents with current density **J**. Show that $\nabla \cdot \mathbf{B} = 0$.
Note: You may use the following identity: $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$.

For a volume charge density $\vec{J}$ over a volume V, Biot-Savart law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \vec{\nabla} \cdot \left[\frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau' \right]$$

$$= \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \cdot \left(\frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

by Leibniz rule (no $\vec{r}$ dependence on the boundary of the integrals)

$$= \frac{\mu_0}{4\pi} \int_V \Big[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \underbrace{(\vec{\nabla} \times \vec{J}(\vec{r}\,'))}_{=0} - \vec{J}(\vec{r}\,') \cdot \Big(\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \Big) \Big] d\tau$$

by the hint.

$=0$ because $\vec{J}(\vec{r}\,')$ does not depend on $\vec{r}$

Let us focus on the term $\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$.

$$\left| \vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right| = \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} \left(\mathscr{r}^2 \frac{1}{\mathscr{r}^2} \right) = \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} (1) = 0$$

Hence $\vec{\nabla} \cdot \vec{B} = 0$ because the whole integrand is 0.