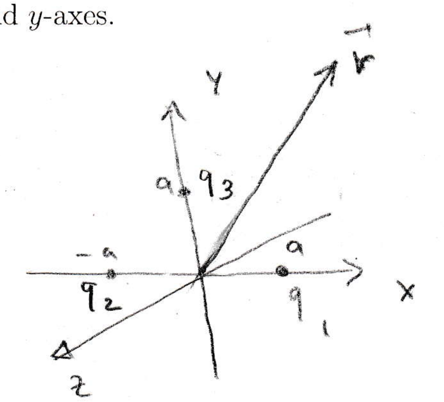
Solution to Midterm Exam Problems

PHYS/ELEC 312

Fall 2019

Problem 1 (15 points) Find the first two non-vanishing terms in the multipole expansion of the electric potential V for the three point charges q_1, q_2, and q_3 with positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ such that $q_1 = q_2$ and $q_3 = -2q_1$, $\mathbf{r}_1 = a\,\hat{\mathbf{x}}$, $\mathbf{r}_2 = -a\,\hat{\mathbf{x}}$, and $\mathbf{r}_3 = a\,\hat{\mathbf{y}}$, where a is a positive real parameter, and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are respectively the unit vectors along the x- and y-axes.

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{|\vec r-\vec r_1|}+\frac{q_2}{|\vec r-\vec r_2|}+\frac{q_3}{|\vec r-\vec r_3|}\right)$$



$$|\vec r-\vec r_j|^2 = r^2 + r_j^2 - 2 r r_j \cos\theta_j$$

$$= r^2\left[1-\frac{2r_j}{r}\cos\theta_j+\left(\frac{r_j}{r}\right)^2\right]$$

$$\frac{1}{|\vec r-\vec r_j|} = \frac{1}{r}\left[1-\frac{2r_j}{r}\cos\theta_j+\left(\frac{r_j}{r}\right)^2\right]^{-1/2}$$

$$= \frac{1}{r}\left[1-\frac{1}{2}\left\{-\frac{2r_j}{r}\cos\theta_j+\left(\frac{r_j}{r}\right)^2\right\}+\frac{3}{8}\left\{-\frac{r_j}{r}\cos\theta_j+\left(\frac{r_j}{r}\right)^2\right\}^2 + \cdots\right]$$

$$= \frac{1}{r}\left[1+\frac{r_j}{r}\cos\theta_j+\left(-\frac{r_j^2}{2r^2}+\frac{3r_j^2\cos^2\theta_j}{8r^2}\right)+\cdots\right]$$

$$= \frac{1}{r}\left[1+\frac{r_j\cos\theta_j}{r}+\frac{(-4+3\cos^2\theta_j)r_j^2}{8r^2}\right]+\cdots$$

$$\boxed{r_1=r_2=r_3=a}\quad,\quad \cos\theta_j = \frac{\vec r\cdot\vec r_j}{r r_j}$$

$$\Rightarrow \cos\theta_1 = \frac{xa}{ra} = \frac{x}{r}\,,\quad \cos\theta_2 = \frac{-xa}{ra} = -\frac{x}{r}$$

$$\cos\theta_2 = \frac{ya}{ra} = \frac{y}{r}$$

$$\Rightarrow V = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_1}{r}\left[1+\frac{ax}{r^2}+\frac{\left(-4+\frac{3x^2}{r^2}\right)a^2}{8r^2}\right]+\right.$$

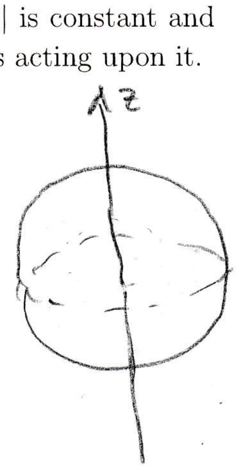
$$+\frac{q_1}{r}\left[1-\frac{ax}{r^2}+\frac{\left(-4+\frac{3x^2}{r^2}\right)a^2}{8r^2}\right]+$$

$$\left.-\frac{2q_1}{r}\left[1+\frac{ay}{r^2}+\frac{\left(-4+\frac{3y^2}{r^2}\right)a^2}{8r^2}\right]+\cdots\right\}$$

$$= \frac{q_1}{4\pi\epsilon_0}\left[\frac{-2ay}{r^3}+\frac{3a^2(x^2-y^2)}{4r^5}+\cdots\right]$$

Problem 2 Consider a homogenous iron sphere of radius R and mass density d that carries a charge Q and a uniform magnetization $\mathbf{M}$ along the z-axis, i.e., $M := |\mathbf{M}|$ is constant and $\mathbf{M} = M\hat{\mathbf{z}}$. The sphere is initially at rest and placed in vacuum with no forces acting upon it.

2.a (5 points) Find the electric field for this configuration.



Inside the sphere:

$\vec{E} = \vec{0}$ because the sphere is a conductor.

Outside the sphere:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

2.b (20 points) Given that the magnetic field outside the sphere is given by

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right),$$

with $m := 4\pi R^3 M/3$, find the angular momentum stored in the electromagnetic field for this configuration.

Note: The Poynting vector and the linear momentum density are respectively given by $\mathbf{S} := \mathbf{E}\times\mathbf{H}$ and $\mathbf{g} := \varepsilon\,\mu\,\mathbf{S}$.

Angular momentum density:

$$\vec{l} = \vec{r}\times\vec{g} = \epsilon\mu\,\vec{r}\times\vec{S} = \epsilon\mu\,\vec{r}\times(\vec{E}\times\vec{H}) = \epsilon\,\vec{r}\times(\vec{E}\times\vec{B})$$

$$\Rightarrow \underline{\vec{l} = \vec{0} \text{ for } r<R}$$

$\underline{\text{For } r>R:}$

$$\vec{E}\times\vec{B} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\times\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

$$= \frac{\mu_0 Q m}{(4\pi)^2\epsilon_0}\frac{\sin\theta}{r^5}\underbrace{\hat{r}\times\hat{\theta}}_{\hat{\phi}} = \frac{\mu_0 Q m}{(4\pi)^2\epsilon_0}\frac{\sin\theta}{r^5}\hat{\varphi}$$

$$\Rightarrow \vec{l} = \frac{\mu_0\epsilon_0 Q m}{(4\pi)^2\epsilon_0}\frac{\sin\theta}{r^4}\hat{r}\times\hat{\varphi}$$

$$\boxed{\hat{r}\times\hat{\varphi}} = (\sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z})\times(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y})$$

$$= \sin\theta\cos^2\varphi\,\hat{z} + \sin\theta\sin^2\varphi\,\hat{z} - \cos\theta\sin\varphi\,\hat{y} - \cos\theta\cos\varphi\,\hat{x}$$

$$= \boxed{\sin\theta\,\hat{z} - \cos\theta(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})}$$

$$\boxed{\vec{L}} = \int_{\mathbb{R}^3}\vec{l}\,d^3\vec{r} = \frac{\mu_0 Q m}{(4\pi)^2}\int_R^\infty dr\,r^2\int_0^\pi d\theta\,\sin\theta\int_0^{2\pi}d\varphi\left\{\frac{\sin\theta}{r^4}\left[\sin\theta\,\hat{z} - \cos\theta(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})\right]\right\}$$

$$= \frac{\mu_0 Q m}{8\pi}\int_R^\infty dr\,\frac{1}{r^2}\int_0^\pi d\theta\,\underbrace{\sin^3\theta}_{\sin\theta(1-\cos^2\theta)}\,\hat{z}$$

$$= \frac{\mu_0 Q m}{8\pi}\underbrace{\left(-\frac{1}{r}\Big|_R^\infty\right)}_{1/R}\Big[\underbrace{\int_0^\pi d\theta\,\sin\theta}_{-\cos\theta|_0^\pi \;=\; 2} - \underbrace{\int_0^\pi d\theta\,\sin\theta\cos^2\theta}_{+\int_1^{-1}du\,u^2 \;=\; \frac{2}{3}}\Big]\hat{z} \qquad \cos\theta = u$$

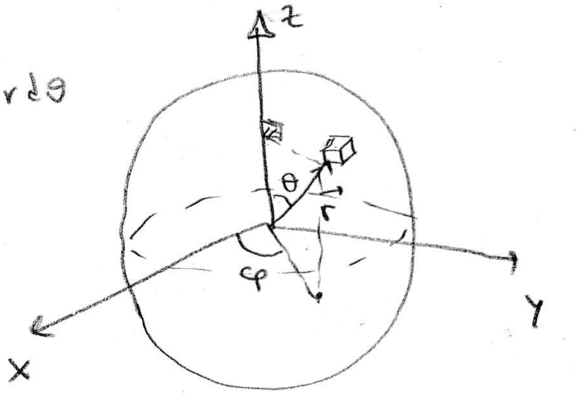
$$= \boxed{\frac{\mu_0 Q m}{6\pi R}\hat{z}}$$

2.c (10 points) Suppose that we gradually heat the sphere so that it undergoes a uniform demagnetization. Show that this causes it to rotate. Find the axis of rotation and the angular velocity ω of the sphere about this axis after it is completely demagnetized.

After demagnetization $M \to 0$, $m \to 0 \Rightarrow \vec{L} \to 0$

By the angular momentum conservation the electromagnetic angular momentum transforms into mechanical angular momentum $\Rightarrow$ $\vec{L}_{mech} \overset{(1)}{=} \dfrac{\mu_0 Q m}{6\pi R}\hat{z} \Rightarrow$

The sphere rotates about $\hat{z}$ axis. To compute its angular frequences we compute:



$$d\vec{L}_{mech} = (\vec{r}\times\vec{v})\, d\, dr^3 = \sin\theta\, r^2 dr\, d\theta$$

$$\vec{v} = (r\sin\theta)\,\omega\,\hat{\varphi}$$

$$\vec{r}\times\hat{\varphi} = r\,\hat{r}\times\hat{\varphi}$$

$$= r\left[\sin\theta\,\hat{z} - \cos\theta\,(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})\right]$$

$$\Rightarrow \vec{L}_{mech} = \int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi\; r^2 \sin\theta \left[\sin\theta\,\hat{z} - \cos\theta\,(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})\right] d\omega$$

$$= 2\pi\, d\omega \underbrace{\int_0^R dr\, r^4}_{\frac{R^5}{5}} \underbrace{\int_0^\pi d\theta\, \sin^3\theta}_{\frac{4}{3}}\,\hat{z} = \frac{8\pi R^5 d\omega}{15}\,\hat{z}$$

$$\Rightarrow \frac{8\pi R^5 d\omega}{15} = \frac{\mu_0 Q m}{6\pi R} \Rightarrow \boxed{\omega = \frac{5\mu_0 Q m}{16\pi^2 d R^6}}$$

Problem 3 (15 points) Consider a stationary, homogeneous, isotropic linear medium with permittivity ε, permeability μ, and conductance σ. Write down the Maxwell's equations and derive the modified wave equation satisfied by every magnetic field $\mathbf{B}$ in this medium.
Hint: You may use the identity, $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$, without proof.

$$\begin{cases} \vec{\nabla} \cdot \vec{D} \overset{①}{=} \rho & \vec{\nabla} \cdot \vec{B} \overset{②}{=} 0 \\ \vec{\nabla} \times \vec{E} \overset{③}{=} -\dfrac{\partial \vec{B}}{\partial t} & \vec{\nabla} \times \vec{H} \overset{④}{=} \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \end{cases}$$

ohm's law: $\vec{J} = \sigma \vec{E}$, $\rho = 0$,

$\vec{D} = \epsilon \vec{E}$, $\vec{H} = \dfrac{1}{\mu} \vec{B}$ $\Rightarrow$

① $\Rightarrow$ $\vec{\nabla} \cdot \vec{E} \overset{⑤}{=} 0$, ④ $\Rightarrow$ $\vec{\nabla} \times \vec{B} \overset{⑥}{=} \mu\sigma \vec{E} + \mu\epsilon \dfrac{\partial \vec{E}}{\partial t}$

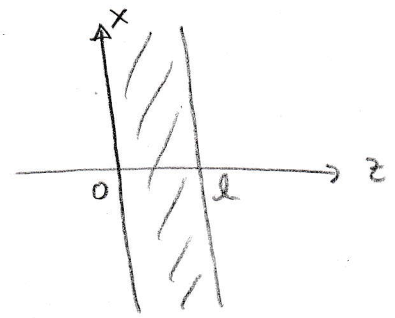
⑥ $\Rightarrow$ $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu\sigma \vec{\nabla} \times \vec{E} + \mu\epsilon \vec{\nabla} \times \dfrac{\partial \vec{E}}{\partial t}$

$$\overset{③}{\Rightarrow} \quad \underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{B})}_{0} - \nabla^2 \vec{B} = \mu\sigma \left(-\frac{\partial \vec{B}}{\partial t}\right) + \mu\epsilon \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 \vec{B} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} - \mu\sigma \frac{\partial \vec{B}}{\partial t} = 0}$$

Problem 4 Consider a slab made of a stationary, homogeneous, and isotropic linear dielectric material with permittivity ε and permeability μ. Suppose that the slab occupies the region $V := \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq \ell\}$, where ℓ is the slab's thickness, and that there is a normally incident monochromatic plane wave with angular frequency ω, polarization vector $\hat{\mathbf{x}}$, and wave vector $k\hat{\mathbf{z}}$ with $k = \omega/c$. A part of this wave is reflected from the boundary of V located at $z = 0$. The rest passes through the slab, exits from its boundary located at $z = \ell$, and propagates towards $z = +\infty$.

4.a (10 points) Write down the expression for the complex electric and magnetic fields modeling the above scenario for $z < 0$, $0 \leq z \leq \ell$, and $z > \ell$ in terms of c, k, ω, and the refractive index $n := \sqrt{\varepsilon\mu/\varepsilon_0\mu_0}$.



$$\vec{E}(z,t) = \begin{cases} [a_1 e^{i(kz-\omega t)} + b_1 e^{i(-kz-\omega t)}]\,\hat{x} & z<0 \\ [a_2 e^{i(k'z-\omega t)} + b_2 e^{i(-k'z-\omega t)}]\,\hat{x} & 0 \leq z \leq l \\ a_3 e^{i(kz-\omega t)}\,\hat{x} & z>l \end{cases}$$

$$\vec{B}(z,t) = \begin{cases} \frac{1}{c}[a_1 e^{i(kz-\omega t)} - b_1 e^{i(-kz-\omega t)}]\,\hat{y} & z<0 \\ \frac{k'}{\omega}[a_2 e^{i(k'z-\omega t)} - b_2 e^{i(-k'z-\omega t)}]\,\hat{y} & 0 \leq z \leq l \\ \frac{1}{c} a_3 e^{i(kz-\omega t)}\,\hat{y} & z>l \end{cases}$$

$$\boxed{k' := \frac{\omega}{v} = \frac{\omega}{\frac{c}{n}} = \frac{n\omega}{c} = nk} \Rightarrow \boxed{\frac{k'}{\omega} = \frac{n}{c}}$$

$$\Rightarrow \vec{E}(z,t) = \begin{cases} [a_1 e^{i(kz-\omega t)} + b_1 e^{i(-kz-\omega t)}]\,\hat{x} & z<0 \\ [a_2 e^{i(nkz-\omega t)} + b_2 e^{i(-nkz-\omega t)}]\,\hat{x} & 0 \leq z \leq l \\ a_3 e^{i(kz-\omega t)}\,\hat{x} & z>l \end{cases}$$

$$\& \quad \vec{B}(z,t) = \begin{cases} \frac{1}{c}[a_1 e^{i(kz-\omega t)} - b_1 e^{i(-kz-\omega t)}]\,\hat{y} & z<0 \\ \frac{n}{c}[a_2 e^{i(nkz-\omega t)} - b_2 e^{i(-nkz-\omega t)}]\,\hat{y} & 0 \leq z \leq l \\ \frac{1}{c} a_3 e^{i(kz-\omega t)}\,\hat{y} & z>l \end{cases}$$

4.b (15 points) Use the matching conditions at $z = 0$ and $z = \ell$ to determine how the amplitude of the complex electric fields for the reflected and transmitted waves are related to the amplitude of the complex electric field for the incident wave.

Note: At the boundaries of V, the parallel components of **E** and **H** fields, and the orthogonal component of **D** and **B** fields are continuous.

At $z=0$: $\vec{E}^{\parallel}$ is continuous $\Rightarrow$ $\boxed{a_1 + b_1 = a_2 + b_2}$ ①

$\vec{H}^{\parallel}$ is continuous $\Rightarrow$ $\frac{1}{\mu_0}\left[\frac{a_1}{c} - \frac{b_1}{c}\right] = \frac{1}{\mu}\left[\frac{n a_2}{c} - \frac{n b_2}{c}\right]$

$\Rightarrow$ $\boxed{a_1 - b_1 = \frac{\mu_0 n}{\mu}(a_2 - b_2)}$ ②

At $z=\ell$: $\vec{E}^{\parallel}$ is continuous $\Rightarrow$

$a_2 e^{inkl} + b_2 e^{-inkl} = a_3 e^{ikl} \Rightarrow$ $\boxed{a_2 e^{i(n-1)kl} + b_2 e^{-i(n+1)kl} = a_3}$ ③

$\vec{H}^{\parallel}$ is continuous $\Rightarrow$ $\frac{1}{\mu}\left(\frac{n}{c}\right)\left(a_2 e^{inkl} - b_2 e^{-inkl}\right) = \frac{1}{\mu_0 c} a_3 e^{ikl}$

$\Rightarrow$ $\boxed{a_2 e^{i(n-1)kl} - b_2 e^{-i(n+1)kl} = \frac{\mu}{\mu_0 n} a_3}$ ④

① & ② $\Rightarrow$ $\begin{cases} a_1 = \frac{1}{2}\left(\frac{\mu_0 n}{\mu} + 1\right) a_2 + \frac{1}{2}\left(1 - \frac{\mu_0 n}{\mu}\right) b_2 \\ b_1 = \frac{1}{2}\left(1 - \frac{\mu_0 n}{\mu}\right) a_2 + \frac{1}{2}\left(1 + \frac{\mu_0 n}{\mu}\right) b_2 \end{cases}$

③ & ④ $\Rightarrow$ $\begin{cases} a_2 = \frac{e^{-i(n-1)kl}}{2}\left(1 + \frac{\mu}{\mu_0 n}\right) a_3 = \frac{\mu}{2\mu_0 n} e^{-i(n-1)kl}\left(\frac{\mu_0 n}{\mu} + 1\right) a_3 \\ \& \\ b_2 = \frac{e^{+i(n+1)kl}}{2}\left(1 - \frac{\mu}{\mu_0 n}\right) a_3 = \frac{\mu}{2\mu_0 n} e^{+i(n+1)kl}\left(\frac{\mu_0 n}{\mu} - 1\right) a_3 \end{cases}$

so let $\boxed{\nu_\pm := \frac{1}{2}\left(1 \pm \frac{\mu_0 n}{\mu}\right)}$ Then

$\Rightarrow \begin{cases} a_1 = \nu_+ a_2 + \nu_- b_2 \\ b_1 = \nu_- a_2 + \nu_+ b_2 \end{cases}$ $\qquad a_2 = \frac{\mu}{\mu_0 n} \nu_+ e^{-i(n-1)kl} a_3$

$\qquad b_2 = \frac{-\mu}{\mu_0 n} \nu_- e^{i(n+1)kl} a_3$

$\Downarrow$

4.b continues :

$$\Rightarrow \quad a_1 = \frac{\mu a_3}{\mu_0 n}\left(\nu_+^2 e^{-i(n-1)kl} - \nu_-^2 e^{i(n+1)kl}\right)$$

$$= \frac{\mu a_3 e^{ikl}}{\mu_0 n}\left(\nu_+^2 e^{-inkl} - \nu_-^2 e^{inkl}\right)$$

$$\Rightarrow \quad \boxed{a_3 = \left[\frac{\mu_0 n}{\mu} e^{-ikl}\left(\nu_+^2 e^{-inkl} - \nu_-^2 e^{inkl}\right)^{-1}\right] a_1}$$

Amplitude for transmitted wave

amplitude for the incident wave

$$b_1 = \frac{\mu a_3}{\mu_0 n}\left(\nu_-\nu_+ e^{-i(n-1)kl} - \nu_+\nu_- e^{i(n+1)kl}\right)$$

$$= \frac{\mu \nu_-\nu_+ e^{ikl} a_3 \left(e^{-inkl} - e^{inkl}\right)}{\mu_0 n}$$

$$\boxed{b_1 = \frac{-2i\mu \nu_-\nu_+ e^{ikl} \sin(nkl)\, a_3}{\mu_0 n}}$$

$$\Downarrow$$

$$b_1 = \left[\frac{-2i\mu\nu_-\nu_+ e^{ikl}\sin(nkl)}{\mu_0 n}\right]\left[\frac{\mu_0 n e^{-ikl}}{\mu}\left(\nu_+^2 e^{-inkl} - \nu_-^2 e^{inkl}\right)^{-1}\right] a_1$$

$$\Rightarrow \quad \boxed{b_1 = \left[\frac{-2i\nu_-\nu_+ \sin(nkl)}{\nu_+^2 e^{-inkl} - \nu_-^2 e^{inkl}}\right] a_1 = \left[\frac{2i\sin(nkl)}{\frac{\nu_-}{\nu_+} e^{inkl} - \frac{\nu_+}{\nu_-} e^{-inkl}}\right] a_1}$$

Amplitude of the reflected wave.

4.c (10 points) Compute the reflection and transmission coefficients for the slab.

$$R = \left|\frac{b_1}{a_1}\right|^2 = \frac{4\sin^2(nlk)}{\left|\frac{\nu_+}{\nu_-} - \frac{\nu_-}{\nu_+} e^{2inkl}\right|^2}$$

$$= \frac{4\sin^2(nkl)}{\left[\frac{\nu_+}{\nu_-} - \frac{\nu_-}{\nu_+}\cos(2nkl)\right]^2 + \frac{\nu_-^2}{\nu_+^2}\sin^2(2nkl)}$$

$$\boxed{R = \frac{2\left[1-\cos(2nkl)\right]}{\left(\frac{\nu_+}{\nu_-}\right)^2 + \left(\frac{\nu_-}{\nu_+}\right)^2 - 2\cos(2nkl)}}$$

$$T = \left|\frac{a_3}{a_1}\right|^2 = \left|\frac{\mu_0 n e^{-ikl}}{\mu\left(\nu_+^2 e^{-inkl} - \nu_-^2 e^{inkl}\right)}\right|^2$$

$$= \frac{\mu_0^2 n^2}{\mu^2 \left|\nu_+^2 - \nu_-^2 e^{2inkl}\right|^2}$$

$$= \frac{\mu_0^2 n^2}{\mu^2 (\nu_+\nu_-)^2 \left|\frac{\nu_+}{\nu_-} - \frac{\nu_-}{\nu_+} e^{2inkl}\right|^2}$$

$$\Rightarrow \boxed{T = \frac{\left(\frac{\mu_0 n}{\mu \nu_+\nu_-}\right)^2}{\left(\frac{\nu_+}{\nu_-}\right)^2 + \left(\frac{\nu_-}{\nu_+}\right)^2 - 2\cos(2nkl)}}$$

$R + T = 1$ ✓