Solutions

Phys/Elec 312: Final Exam

December 30, 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

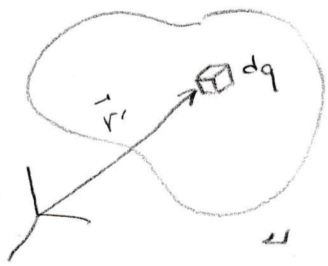
- You have 2 hours and 45 minutes.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.

Problem 1 Derive the statement of the Poynting theorem and conservation of EM energy in vacuum using the following steps.

1.a (10 points) Write down Maxwell's equation in vacuum for a system with charge and current distributions ρ and $\mathbf{J}$ confined to a region $\mathcal{V}$ of space, and show that the rate of change of the work W done by the EM fields on the changes is given by $\frac{dW}{dt} = \int_{\mathcal{V}} \mathbf{E}\cdot\mathbf{J}\, d^3\mathbf{r}$.

$$\begin{cases} \vec\nabla\cdot\vec E = \frac{\rho}{\epsilon_0}, & \vec\nabla\cdot\vec B = 0 \\ \vec\nabla\times\vec E = -\frac{\partial}{\partial t}\vec B, & \vec\nabla\times\vec B = \mu_0\vec J + \epsilon_0\mu_0\frac{\partial\vec E}{\partial t} \end{cases}$$

$$\frac{d}{dt}\delta W = \frac{d}{dt}\delta\vec F\cdot d\vec x = \delta\vec F\cdot\vec v$$



$$= \delta q(\vec E + \vec v\times\vec B)\cdot\vec v$$

$$= \rho\,\underbrace{\delta V}_{\text{volume elem}}\,\vec E\cdot\vec v$$

$$= \vec E\cdot(\rho\vec v)\,\delta V$$

$$= \vec E\cdot\vec J\;\delta V$$

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$$\frac{d}{dt}W = \frac{d}{dt}\int_\nu \delta W = \int_\nu \frac{d}{dt}\delta W = \int_\nu \vec E\cdot\vec J\, \underset{\overset{\shortparallel}{d^3\vec r}}{dV}.$$

1.b (10 points) Use Maxwell's equations and the identity, $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$, to express $\mathbf{E}\cdot\mathbf{J}$ in terms of the energy density $u := \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})$ and the Poynting vector $\mathbf{S} := \mathbf{E}\times\mathbf{H}$.

$$\vec{E}\cdot\vec{J} = \vec{E}\cdot\left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \frac{1}{\mu_0}\underbrace{\vec{E}\cdot(\vec{\nabla}\times\vec{B})}_{-\vec{\nabla}\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot(\vec{\nabla}\times\vec{E})} - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} \to -\frac{\partial B}{\partial t}$$

$$= -\underbrace{\vec{\nabla}\cdot\left(\frac{1}{\mu_0}\vec{E}\times\vec{B}\right)}_{\vec{S}} + \underbrace{\frac{1}{\mu_0}\vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{B}}{\mu_0} + \epsilon_0\vec{E}\cdot\vec{E}\right)}$$

$$-\frac{1}{2}\frac{\partial}{\partial t}\underbrace{\left(\frac{\vec{B}\cdot\vec{B}}{\mu_0} + \epsilon_0\vec{E}\cdot\vec{E}\right)}_{-\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right]}$$

$$-\frac{\partial}{\partial t}\underbrace{\left[\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right]}_{u}$$

$$\Rightarrow \boxed{\vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}u - \vec{\nabla}\cdot\vec{S}}$$

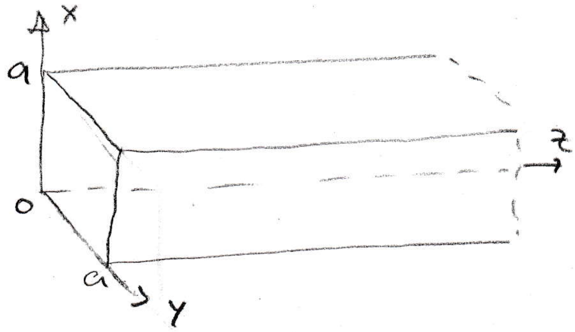
1.c (5 points) Show that if the EM fields do not do any work on the charges, then u and $\mathbf{S}$ satisfy the continuity equation for energy. Give a derivation of this equation.

$$\frac{dW}{dt} = \int_{\mathcal{V}}\left(-\frac{\partial}{\partial t}u - \vec{\nabla}\cdot\vec{S}\right)d^3\vec{r}$$

If $W = 0 \Rightarrow \int_{\mathcal{V}}\left(-\frac{\partial}{\partial t}u - \vec{\nabla}\cdot\vec{S}\right)d^3\vec{r}' = 0$ for every $\mathcal{V}$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}u + \vec{\nabla}\cdot\vec{S} = 0}$$

Problem 2 (20 points) Consider a rectangular waveguide with a square cross section of side length a. The wave equation for monochromatic TE waves of frequency ω that propagate in this waveguide reduces to $\left(\partial_x^2 + \partial_y^2 + \omega^2/c^2 - k^2\right) B_z(x, y) = 0$, where k is the wavenumber, and the boundary conditions $\partial_x B_z = \partial_y B_z = 0$ at the boundaries of the waveguide, i.e., $x = 0, a$ and y arbitrary, and $y = 0, a$ and x arbitrary. Show that this waveguide has a finite number of propagating modes. Find the values of the frequency where this number is 2.



$$B_z = X(x)\, Y(y)$$

$$\partial_x B_z\Big|_{\partial} = 0 \implies X'(0) = X'(a) = 0$$

$$\partial_y B_z\Big|_{\partial} = 0 \implies Y'(0) = Y'(a) = 0$$

$$\underbrace{\frac{X''}{X}}_{-k_x^2} + \underbrace{\frac{Y''}{Y}}_{-k_y^2} + \frac{\omega^2}{c^2} - k^2 = 0 \qquad (*)$$

$$\implies X = \alpha \cos(k_x x) + \beta \sin(k_x x)$$

$$\implies X' = k_x\left[-\alpha \sin(k_x x) + \beta \cos(k_x x)\right]$$

$\implies \beta = 0$

$k_x a = \pi m \implies k_x = \dfrac{\pi m}{a}$

$\underbrace{m \in \{0, 1, 2, \dots\}}_{\mathbb{N}}$

$$\implies X = \alpha \cos\left(\frac{\pi m x}{a}\right)$$

Similarly we find $k_y = \dfrac{\pi n}{a}$ & $Y = \gamma \cos\left(\dfrac{\pi n y}{a}\right)$ for some $n \in \mathbb{N}$

$\Downarrow$

$$B_z = \delta \cos\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{a}\right) \quad \text{for some } \delta \in \mathbb{C}.$$

$$\text{Also } (*) \implies \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k^2 = \frac{\pi^2}{a^2}(m^2 + n^2) + k^2$$

$$\implies k^2 = \frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}(m^2 + n^2) \geq 0$$

$$\implies \omega > \frac{\pi c}{a}\sqrt{m^2 + n^2} =: \omega_{mn}$$

$$\text{So } m^2 + n^2 \in 1, 2 \implies \omega_{11} \leq \omega < \omega_{02}$$

$$\boxed{\frac{\sqrt{2}\,\pi c}{a} \leq \omega < \frac{2\pi c}{a}}$$

Problem 3 (15 points) Two particles of mass m_1 and m_2 and initial velocities $v_1\hat{\mathbf{x}}$ and $-v_2\hat{\mathbf{x}}$ collide and stick together to create a particle of mass M. Determine the maximum value of M and find conditions on m_1, m_2, v_1, and v_2 under which M attains this value.

Energy conservation: $E_1 + E_2 = E$

$$m_1c^2\gamma_1 + m_2c^2\gamma_2 = Mc^2\gamma \quad (1)$$

$$\gamma_i := \frac{1}{\sqrt{1-\beta_i^2}}, \quad \gamma := \frac{1}{\sqrt{1-\beta}} \qquad \beta_i := \frac{v_i}{c}, \quad \beta := \frac{V}{c})$$

Speed of M

M is maximum if $\gamma = 1$

$\Rightarrow V = 0 \Rightarrow$ the created particle is at rest.

In this case (1) $\Rightarrow$ $\boxed{m_1\gamma_1 + m_2\gamma_2 = M}$

Momentum conservation: $m_1\gamma_1\vec{v}_1 + m_2\gamma_2\vec{v}_2 = \vec{0}$

$$\Downarrow$$

$$\boxed{m_1\gamma_1 v_1 - m_2\gamma_2 v_2 = 0}$$

$$\Rightarrow \frac{m_1^2\beta_1^2}{1-\beta_1^2} = \frac{m_2^2\beta_2^2}{1-\beta_2^2} \Rightarrow \left(\frac{m_1}{m_2}\right)^2\left(\frac{1}{\beta_1^{-2}-1}\right) = \frac{1}{\beta_2^{-2}-1}$$

$$\Rightarrow \beta_2^{-2} - 1 = \left(\frac{m_2}{m_1}\right)^2(\beta_1^{-2}-1)$$

$$\Rightarrow \beta_2^{-2} = \left(\frac{m_2}{m_1}\right)^2(\beta_1^{-2}-1) + 1$$

$$\Rightarrow \boxed{v_2 = \beta_2 c} = \frac{c}{\sqrt{\left(\frac{m_2}{m_1}\right)^2(\beta_1^{-2}-1)+1}} = \boxed{\frac{m_1 c}{m_2\sqrt{\frac{c^2}{v_1^2} - 1 + \frac{m_1^2}{m_2^2}}}}$$

Problem 4 (15 points) Let $\mathcal{O}$, $\tilde{\mathcal{O}}$, and $\mathcal{O}'$ be inertial frames with associated Cartesian spacetime coordinates, (x^0, x^1, x^2, x^3), $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$, and (x'^0, x'^1, x'^2, x'^3), respectively. Suppose that $\tilde{\mathcal{O}}$ is at rest with respect to $\mathcal{O}$ but their space coordinates are related via a rotation by an angle θ about their common x^3- and $\tilde{x}^3$-axes, so that

$$\tilde{x}^0 = x^0, \qquad \tilde{x}^1 = \cos\theta\, x^1 - \sin\theta\, x^2, \qquad \tilde{x}^2 = \sin\theta\, x^1 + \cos\theta\, x^2, \qquad \tilde{x}^3 = x^3,$$

and $\mathcal{O}'$ moves with constant velocity $\mathbf{v} = v\,\hat{\tilde{\mathbf{x}}}^1$ with respect to $\tilde{\mathcal{O}}$ such that

$$x'^0 = \gamma(\tilde{x}^0 - \beta\tilde{x}^1), \qquad x'^1 = \gamma(\tilde{x}^1 - \beta\tilde{x}^0), \qquad x'^2 = \tilde{x}^2, \qquad x'^3 = \tilde{x}^3,$$

where $\beta := v/c$ and $\gamma := (1-\beta^2)^{-1/2}$. Find the Lorentz transformation matrices $\tilde{\mathbf{\Lambda}}$, $\mathbf{\Lambda}'$, and $\mathbf{\Lambda}$ satisfying $\tilde{x}^\mu = \tilde{\Lambda}^\mu{}_\nu\, x^\nu$, $x'^\mu = \Lambda'^\mu{}_\nu\, \tilde{x}^\nu$, and $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$.

$$\tilde{\Lambda}\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{bmatrix} \Rightarrow \tilde{\Lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda'\begin{bmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{bmatrix} = \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} \Rightarrow \Lambda' = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda = \Lambda'\tilde{\Lambda} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\gamma\beta\cos\theta & \gamma\beta\sin\theta & 0 \\ -\gamma\beta & \gamma\cos\theta & -\gamma\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5 Let $\mathcal{O}$ and $\mathcal{O}'$ be the inertial frames described in Problem 4. Suppose that the observer in $\mathcal{O}$ observes a perfect electric dipole with scalar potential given by, $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$, where $\mathbf{r} := x^1\hat{\mathbf{x}}^1 + x^2\hat{\mathbf{x}}^2 + x^2\hat{\mathbf{x}}^2$ is the position vector, $r := |\mathbf{r}|$, $\hat{\mathbf{r}} := \mathbf{r}/r$, and $\mathbf{p}$ is the dipole moment.

5.a (5 points) Calculate the electric field for this system as measured in the reference frame $\mathcal{O}$. Give the details of this calculation.

$$\vec{E} = -\vec{\nabla} V = -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{3}\hat{x}^j\,\partial_j\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\sum_{i,j=1}^{3}\hat{x}^j\,p_i\,\underbrace{\partial_j\left(\frac{x^i}{r^3}\right)}_{\frac{\delta_{ij}}{r^3} + x^i\,\underbrace{\partial_j\left[\sum_{k=1}^{3}(x^k)^2\right]^{-3/2}}_{-\frac{3}{2}r^{-5}\,2x^j}}$$

$$\underbrace{\frac{\delta_{ij}}{r^3} + x^i\left(-\frac{3}{2}r^{-5}\,2x^j\right)}_{\frac{\delta_{ij}}{r^3} - \frac{3x^i x^j}{r^5}}$$

$$= -\frac{1}{4\pi\epsilon_0}\sum_{i,j=1}^{3}\left(\frac{\delta_{ij}\,\hat{x}^j p_i}{r^3} - \frac{3\hat{x}^j x^j p_i x^i}{r^5}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}}{r^3} - \frac{3\vec{r}\,(\vec{p}\cdot\vec{r})}{r^5}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{r})\,\hat{r} - \vec{p}}{r^3}\right]$$

5.b (20 points) Determine the component of the magnetic field along the x'^1-axis as measured in the reference frame $\mathcal{O}'$.

Hint: The contravariant EM field tensor $F^{\mu\nu}$ is related to the electric and magnetic fields according to

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{bmatrix}.$$

$$B'_1 = F'^{23} = \Lambda^2{}_\mu \Lambda^3{}_\nu F^{\mu\nu}$$

According to the sol. of Problem 4:

$$\Lambda^2{}_0 = \Lambda^2{}_3 = \Lambda^3{}_0 = \Lambda^3{}_1 = \Lambda^3{}_2 = 0$$

$$\Rightarrow B'_1 = \Lambda^2{}_1 \Lambda^3{}_3 \underbrace{F^{13}}_{B_2 = 0} + \Lambda^2{}_2 \Lambda^3{}_3 \underbrace{F^{23}}_{B_1 = 0} = 0.$$