**SOLID STATE PHYSICS HW#6**

**Question 1. Kinetic energy and pressure of electron gas (Kittel 6.1-2)**

**(a)** Consider a 3-$d$ gas of N electrons at absolute zero. Since the model consists of free electrons, the internal energy is equal to total kinetic energy of the system. At 0 K the system is in its ground state, that is, the states above the state which has the Fermi energy $\epsilon_f$ are empty. Then the Fermi-Dirac distribution function (the probability that an orbital at energy $\epsilon$ will be occupied in an ideal electron gas in thermal equilibrium) is equal to a step function at zero temperature, namely 1 for $\epsilon \leq \epsilon_f$ and 0 for $\epsilon > \epsilon_f$ (See Kittel, Ch.6, **Heat Capacity of the Electron Gas**). The kinetic energy is then given by

$$U = \int_0^{\epsilon_f} d\epsilon \epsilon D(\epsilon)\,, \tag{1}$$

where $D(\epsilon)$ is density of states. The number of states $N(K)$ for a wave vector $\vec{K}$ is given by

$$N(K) = \frac{2\frac{4}{3}\pi K^3}{\left(\frac{2\pi}{L}\right)^3} = \frac{K^3 V}{3\pi^2} \tag{2}$$

In order to find $N(\epsilon)$, the energy is $\epsilon_n = \frac{\hbar^2 K_n^2}{2m}$. Then, $N(\epsilon)$ and $D(\epsilon)$ are

$$\begin{aligned} N(\epsilon) &= \frac{V}{3\pi^2}\left(\frac{2m\epsilon}{\hbar^2}\right)^{3/2}\,, \\ D(\epsilon) &= \frac{dN}{d\epsilon} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}\,, \end{aligned} \tag{3}$$

respectively. Using the above expression for density of states $D(\epsilon)$ in eq. (1) gives the kinetic energy for absolute zero, which is $U = \frac{3}{5}N\epsilon_f$.

**(b)** The pressure is given by $P = -\frac{dU}{dV}$. Using the expression for the kinetic energy in **(a)** and the fermi energy $\epsilon_f = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$,

$$U = \frac{3}{5}N\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}\,. \tag{4}$$

Ultimately, $P = -\frac{dU}{dV}$ is equal to $\frac{2U}{3V}$.

**Question 2. Chemical Potential in $2D$ (Kittel 6.3)**

In two dimensions, $N(K) = \frac{2\pi K^2}{\left(\frac{2\pi}{L}\right)^2}$, where $K^2 = \frac{2m\epsilon}{\hbar^2}$, so $N(\epsilon) = \frac{m\epsilon}{\pi\hbar^2}L^2$. Density of states $D(\epsilon)$ is given by

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{mL^2}{\pi\hbar^2}\,. \tag{5}$$

In the question, this result is given as a hint (note that you should multiply the expression in the hint with the area $L^2$, in order to reach the result above). For finite temperatures the number of electrons is equal to

$$N = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon) = \frac{mL^2}{\pi\hbar^2}\int_0^\infty d\epsilon \frac{1}{e^{(\epsilon-\mu)/k_BT}+1}\,. \tag{6}$$

It is easy to evaluate the integral when one multiplies both the numerator and denominator with $e^{-(\epsilon-\mu)/k_BT}$, namely

$$N = \frac{mL^2}{\pi\hbar^2}\int_0^\infty d\epsilon \frac{e^{-(\epsilon-\mu)/k_BT}}{1+e^{-(\epsilon-\mu)/k_BT}}\,. \tag{7}$$

Defining $x \equiv e^{-(\epsilon-\mu)/k_BT}+1$, the integral is rewritten as

$$N = \frac{mL^2}{\pi\hbar^2}k_BT\int_1^{e^{\mu/k_BT}+1}\frac{dx}{x}\,. \tag{8}$$

Then N is equal to

$$N = \frac{mL^2}{\pi\hbar^2}k_BT\ln\left(e^{\mu/k_BT}+1\right). \tag{9}$$

The chemical potential $\mu$ is obtained as

$$\mu = k_BT\ln\left(e^{n\pi\hbar^2/mk_BT}-1\right) \tag{10}$$

where $n = \frac{N}{L^2}$.

**Question 3. Fermi gases in astrophysics (Kittel 6.4)**

**(a)** The Sun consists of approximately 75% Hydrogen and 25% Helium. The molar mass of He and H is nearly $4g$ and $1g$, respectively. One H includes 1 electron, whereas one He has 2 electrons. So, for instance, in $7g$ of the Sun, there are $5N_a$ electrons, where $N_a$ is Avogadro's number. That is, in $2\times10^{33}g$ there are approximately $10^{57}$ electrons. Calculating the Fermi energy (in 3 dimensions),

$$\epsilon_f = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}\,, \tag{11}$$

where $m_e$ is the mass of an electron, and $V = \frac{4}{3}\pi R^3$ ($R = 2 \times 10^9$cm), one finds that it is nearly equal to $3,5 \times 10^4$eV.

**(b)** In the relativistic limit, the Fermi energy is given by $\epsilon_f \simeq \hbar k_f c$. $k_f$ is proportional with $\left(\frac{N}{V}\right)^{1/3}$. One finds that

$$\epsilon_f \simeq \hbar c \left(\frac{N}{V}\right)^{1/3} . \tag{12}$$

**(c)** "Pulsars are highly magnetized rotating neutron stars which emit a beam of detectable electromagnetic radiation in the form of radio waves" (Wikipedia). So their Fermi energy should be evaluated in the relativistic limit. From **(a)**, $N_e \simeq 10^{57}$, and $V = \frac{4\pi}{3} \times 10^{12} m^3$. Setting these numerical values and the other constants into eq. (12), $\epsilon_f$ is found to be nearly $10^8$eV.