## §13.2 Extreme Values of Functions Defined on Restricted Domains

Recall that a continous function $f(x, y)$ defined on a closed and bounded region $D$ in $\mathbb{R}^2$ takes its absolute maximum/minimum on $D$. In this section we will see how to find these absolute extreme values.

**Theorem 1.** Suppose that a function $f(x, y)$ has a local or absolute extreme value at some point $(a, b)$ in its domain. Then $(a, b)$ is one of the following:

i) a critical point, i.e. $\nabla f(a, b) = 0$

ii a singular point, i.e. $\nabla f(a, b)$ does not exists

iii) a boundary point.

So the theorem provides a useful method to find absolute extreme values:

Step 1: Find all critical and singular points of $f$.

Step 2: Find the extreme values of $f$ on the boundary.

Step 3: Compare the values of $f$ at the boundary points and at critical and singular points.

**Example 1.** Find the absolute maximum and minimum values of $f(x, y) = x^2 + 2xy - y^2$ on the disc $x^2 + y^2 \leq 1$.

*Solution:* Since $f(x, y)$ is continuous on the unit disc, $f$ takes its absolute extreme values. First we compute the partial derviatives of $f$ as

$$f_1(x, y) = 2x + y, \;\; f_2(x, y) = x - 2y.$$

So $f$ has no singular point and the only critical point is $(0, 0)$.

Now we consider the boundary points, i.e. the points on the unit circle $x^2 + y^2 = 1$. It is enough to find the absolute extreme values of $f$ on the boundary as we will compare them with $f(0, 0)$.

It is useful to parametrize the boundary whenever it is possible. We can simply take the parametrization

$$x = \cos t, \;\; y = \sin t, \;\; t \in [0, 2\pi].$$

Then we have

$$f(\cos t, \sin t) = \cos^2 t + 2 \cos t \sin t - \sin^2 t = \cos(2t) + \sin(2t)$$

So we are reduced to computing the absolute extreme values of

$$g(t) = \cos(2t) + \sin(2t), \ t \in [0, 2\pi]$$

The absolute maximum and minimum on the boundary are

$$g(\pi/8) = \sqrt{2} \text{ and } g(5\pi/8) = -\sqrt{2}$$

respectively (Exercise). Since $f(0, 0) = 0$ we conclude that the absolute maximum and minimum of $f(x, y)$ on $x^2 + y^2 \leq 1$ are

$$f(\cos(\pi/8), \sin(\pi/8)) = \sqrt{2}, \quad \text{and} \quad f(\cos(5\pi/8), \sin(5\pi/8)) = -\sqrt{2}$$

**Example 2.** Find the absolute maximum and minimum values of $f(x,y) = xy + x^2$ on the triangular region $D$ given by $-1 \leq x \leq 1, x-1 \leq y \leq -x+1$.

*Solution:* First by computing the partial derviatives $f_1(x,y) = y+2x$, $f_2(x,y) = x$ we see that $f$ has no singular points, and also that the only critical point is $(0,0)$.

The boundary of the domain is the triangle with vertices $(1,0)$, $(-1,-2)$ and $(-1,2)$. So the sides of the triangle can be parametrized by the line segments

$$x = -1, -2 \leq y \leq 2$$

$$y = -x+1, -1 \leq x \leq 1$$

$$y = x-1, -1 \leq x \leq 1$$

So on each side of the triangle our function $f$ reduces to a function in single variable as

$$g_1(y) = -y+1, -2 \leq y \leq 2$$

$$g_2(x) = x, -1 \leq x \leq 1$$

$$g_3(x) = 2x^2 - x, -1 \leq x \leq 1$$

If we compute the extreme values of the above functions on the specified domains, we see that the maximum and minimum on the boundary are $f(-1,-2) = 3$ and $f(-1,2) = -1$ respectively (Exercise). Since $f(0,0) = 0$, the absolute maximum and minimum of $f(x,y)$ on $D$ are $f(-1,-2) = 3$ and $f(-1,2) = -1$ respectively.